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1988 J. Phys. A: Math. Gen. 21 L893

(http://iopscience.iop.org/0305-4470/21/18/005)

J. Phys. A: Math. Gen. 21 (1988) L893-L897. Printed in the UK

LETTER TO THE EDITOR

Longitudinal size exponent for two-dimensional directed animals

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Received 8 June 1988

Abstract. Using phenomenological renormalisation with strip widths up to size 23, we have numerically estimated the longitudinal size exponent $\nu_{\parallel}$ for two-dimensional directed animals. We get $\nu_{\parallel} = 0.817\,33 \pm 0.000\,05$. This rules out the simple fraction 9/11 suggested by earlier numerical data. The effective correction to scaling exponent in the continuous-time version of the problem is 1.5 ± 0.1.

The directed animals problem in two dimensions is one of the simplest models showing a non-trivial critical relaxation to equilibrium, and has generated much interest in recent years. Exact expressions for the number of distinct animals with n sites and their average width have been obtained by establishing an equivalence to a hard-square lattice gas with next-nearest-neighbour interactions at a disorder point [1, 2], to a one-dimensional lattice gas with nearest-neighbour exclusion [3], to a subset of random walks in one dimension [4, 5] and by finding the largest eigenvector of the transfer matrix [6, 7]. None of these approaches have succeeded in determining exactly the longitudinal size exponent $\nu_{\parallel}$, defined by the relation $\langle R_{\parallel,n} \rangle \sim n^{\nu_{\parallel}}$, where $\langle R_{\parallel,n} \rangle$ is the average extent in the 'preferred' direction of a directed animal having n sites. By extrapolating the exact values of the average caliper extent for small n, Redner and Yang [8] obtained the estimate $\nu_{\parallel} = 0.800 \pm 0.001$. Analysing a somewhat longer series for the radius of gyration, Privman and Barma [9] estimated $\nu_{\parallel} = 0.8177 \pm 0.0012$. The latter result also agrees with the phenomenological renormalisation calculation of Nadal *et al* [6] who obtained $\nu_{\parallel} = 0.8185 \pm 0.0010$, and noted that it is well approximated by the simple fraction 9/11.

In this letter, we report the result of a phenomenological renormalisation calculation to estimate $\nu_{\parallel}$ numerically. We study the equivalent problem of critical slowing down at the Lee-Yang edge singularity. At the critical point, the relaxation time on a ring on N sites varies as N^z, where z is the dynamical critical exponent $= \nu_{\parallel}/\nu_{\perp}$. We determined the relaxation rates by numerically diagonalising the stochastic matrices for $N = 2$-23, and knowing the exact value $\nu_{\perp} = 1/2$, we find

$$\nu_{\parallel} = 0.817\,33 \pm 0.000\,05. \tag{1}$$

This value lies roughly within the error bars quoted in [6, 9], but our error bars are an order of magnitude smaller. The simple fraction 9/11, which was suggested as a possible exact value by earlier data, seems to be clearly ruled out. We have also estimated the correction to scaling exponent Ω, and find that it is larger than 1, with a value ≈ 1.5.

0305-4470/88/180893+05$02.50

We study a continuous-time Markovian evolution of a one-dimensional lattice gas with nearest-neighbour exclusion on a ring of N sites. The dynamics is single-spin-flip Glauber type with transition rates defined as follows. An empty site with both neighbours empty has a probability $z\,\mathrm{d}t$ of becoming occupied in a subsequent small time interval $\mathrm{d}t$. An occupied site has a probability $\mathrm{d}t$ of becoming empty in the same interval.

These transition rates clearly do not conserve the particle number, but satisfy the detailed balance condition. The steady state corresponds to the thermodynamic state of a lattice gas with nearest-neighbour exclusion at activity z. The exact transcription of the directed site animals on a square lattice to a Markovian evolution of a one-dimensional gas gives a *discrete-time* dynamics, with odd and even sites updated alternately. Choosing a *continuous-time* dynamics results in a much sparser transition matrix, and removes the restriction that N be even. This is very important as the largest value of N reachable is limited by computer capacity, and with only even N allowed, our generated series would be only half as long, with resulting extrapolations much more uncertain. The change from discrete to continuous time is not expected to change the universality class of the problem.

Let $P(C, t)$ be the probability of the configuration C of occupied sites on the ring at time t. Then $P(C, t)$ satisfies a master equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}t} P(C, t) = -\sum_{C'} W(C, C')P(C', t) \tag{2}$$

where $W(C, C')$ is the stochastic rate matrix. The lowest eigenvalue of W is zero, corresponding to the steady state. Let the second-lowest eigenvalue of W be $\lambda_N(z)$. All time-dependent connected correlation functions of the system decay as $\exp(-\lambda_N(z)t)$ for large times t. For real positive values of z, $\lambda_N(z)$ stays finite even as N tends to infinity, consistent with the absence of phase transitions in one dimension. However, the matrix elements of W are simple linear functions of z and the matrix can be analytically continued to negative values of z.

The steady-state properties of the system are easily determined from the partition function $\Omega_N(z)$. It is easy to show that

$$\Omega_N(z) = \mathrm{Tr}\begin{pmatrix} 1 & 1 \\ z & 0 \end{pmatrix}^N. \tag{3}$$

The zeros of $\Omega_N(z)$ lie on the negative real z axis. In the limit of large N, the zeros extend over the real time $-\infty \leqslant z \leqslant -1/4$. The point $z_c = -1/4$ is a critical point of the problem corresponding to the Lee–Yang edge singularity. From finite-size scaling theory it is expected that for large N

$$\lambda_N(z_c) = AN^{-2\nu_\parallel}(1 + BN^{-\Omega} + \text{higher-order terms}) \tag{4}$$

where A and B are some constants and Ω is the correction to scaling exponent.

For finite N, it is straightforward to determine $\lambda_N(z_c)$ numerically. The number of allowed configurations of the gas is Ω_N $(z = 1)$, which increases as $(1.618\ldots)^N$ for large N. Working in a sector invariant under translations and reflection, one is able to reduce the dimension of the matrix W by a factor $\leqslant 2N$. The dimension is further decreased by 1 by knowing the exact left and right eigenvectors corresponding to the eigenvalue 0. We used a simple relaxation algorithm. Numerically, the left eigenvector of W is better behaved, and easier to determine than the right eigenvector. We start

with an arbitrary initial (left) vector $\psi(C)$, and iterate it using the formula

$$\psi'(C)=\left(\psi(C)-(\Delta t)\sum_{C'}\psi(C')\,W(C',C)\right)\frac{1}{F} \tag{5}$$

where Δt is a small time step, and F is a normalisation constant chosen so that $\|\psi'(C)\|=1$. After a large number of iterations $\psi(C)$ converges to the left eigenvector, and $(1-F)/\Delta t$ converges to the smallest (non-zero) eigenvalue $\lambda_N(z)$.

To control the accumulated numerical errors, we stored $4W$ as an integer array (as $z_c=-\frac{1}{4}$), and elements of $\psi(C)$ in double precision (96 bits). The iteration was stopped only when successive λ_N values converged to 19 decimal places. The resulting values of $\lambda_N(z_c)$ are listed in table 1 for $N=2$–23.

To analyse this series we have used a six-term sequential fit of the form

$$\log\lambda_N(z_c)=A_0N+A_1+A_2/N^{-A_4}+A_3/N^{-A_5}. \tag{6}$$

The six unknown parameters $A_0,\ldots,A_5$ are determined using six successive values of $\lambda_N(z_c)$. The results are displayed in table 2. We see that the values of A_0 show a good convergence. The exponent A_5 tends to increase for large N along with the amplitude A_3, indicating that the form of the second correction term is perhaps not appropriate. However, the value of A_0 is not very sensitive to the fitting values A_3 and A_5.

In table 3, we show the values of five-term sequential fits if we fix $A_4=1.5$ in (6) exactly. The convergence is seen to be very good and we conclude

$$A_0=-1.634\,65\pm0.000\,10 \tag{7}$$

Table 1. Values of the critical relaxation rate $\lambda_N(z=z_c)$ for various values of N. The size of the matrix diagonalised is d_N.

N	d_N	λ_N
2	1	0.500 000 000 000 000
3	1	0.250 000 000 000 000
4	2	0.156 929 669 182 746
5	2	0.109 611 796 797 792
6	4	0.081 732 005 752 465
7	4	0.063 737 470 802 681
8	7	0.051 363 960 620 292
9	8	0.042 446 461 629 277
10	13	0.035 781 687 538 384
11	15	0.030 653 799 048 695
12	25	0.026 613 590 856 938
13	30	0.023 366 739 668 568
14	48	0.020 713 390 145 382
15	63	0.018 513 690 439 375
16	98	0.016 667 230 147 525
17	132	0.015 100 297 537 858
18	208	0.013 757 720 261 048
19	290	0.012 597 490 278 604
20	454	0.011 587 133 430 694
21	656	0.010 701 202 545 236
22	1021	0.009 919 511 886 819
23	1509	0.009 225 871 493 041

Table 2. Values of the constants A_0 to A_5 obtained by the sequential fit to (6) using the values λ_N to λ_{N+5}.

N	A_0	A_1	A_2	A_3	A_4	A_5
7	−1.634 481 79	0.441 453	−0.315 370	1.643 77	1.569 89	3.814 41
8	−1.634 527 35	0.441 640	−0.306 575	1.773 83	1.554 88	3.899 82
9	−1.634 556 58	0.441 762	−0.300 294	1.929 84	1.544 20	3.981 15
10	−1.634 577 46	0.441 852	−0.295 320	2.129 87	1.535 79	4.065 62
11	−1.634 593 43	0.441 921	−0.291 141	2.399 46	1.528 77	4.157 98
12	−1.634 606 17	0.441 978	−0.287 517	2.778 22	1.522 71	4.262 10
13	−1.634 616 51	0.442 024	−0.284 351	3.327 09	1.517 45	4.380 61
14	−1.634 625 08	0.442 063	−0.281 551	4.163 02	1.512 81	4.517 91
15	−1.634 632 21	0.442 096	−0.279 093	5.493 51	1.508 74	4.677 10
16	−1.634 638 18	0.442 123	−0.276 932	7.764 76	1.505 16	4.864 06
17	−1.634 643 22	0.442 147	−0.275 039	11.987 34	1.502 03	5.085 70
18	−0.634 647 46	0.442 167	−0.273 391	20.785 01	1.499 30	5.351 84

Table 3. Values of the constants A_0 to A_5 obtained by sequential fit of data to the form (6) using the values λ_N to λ_{N+4}, fixing $A_4 = 1.5$ exactly.

N	A_0	A_1	A_3	A_2	A_5
2	−1.637 829 60	0.451 333 84	1.191 139	−0.350 662	3.375 73
3	−1.636 076 17	0.446 507 97	1.266 586	−0.316 508	3.554 03
4	−1.635 253 91	0.444 104 58	1.439 035	−0.296 328	3.752 50
5	−1.634 949 81	0.443 172 75	1.650 911	−0.287 299	3.907 80
6	−1.634 818 78	0.442 755 01	1.895 839	−0.282 683	4.035 96
7	−1.634 752 39	0.442 536 07	2.189 518	−0.279 957	4.151 71
8	−1.634 714 36	0.442 406 94	2.558 593	−0.278 167	4.263 85
9	−1.634 690 77	0.442 324 78	3.038 374	−0.276 913	4.376 97
10	−1.634 675 43	0.442 270 15	3.674 520	−0.276 003	4.493 01
11	−1.634 665 20	0.442 232 95	4.525 697	−0.275 333	4.612 19
12	−1.634 658 29	0.442 207 39	5.664 507	−0.274 837	4.733 50
13	−1.634 653 62	0.442 189 86	7.170 603	−0.274 473	4.854 70
14	−1.634 650 51	0.442 177 99	9.110 715	−0.274 211	4.972 46
15	−1.634 648 49	0.442 170 18	11.478 987	−0.274 028	5.081 69
16	−1.634 647 26	0.442 165 34	14.104 036	−0.273 907	5.175 69
17	−1.634 646 60	0.442 162 70	16.513 024	−0.273 838	5.245 49
18	−1.634 646 35	0.442 161 73	17.880 794	−0.273 811	5.279 80
19	−1.634 646 43	0.442 162 05	17.268 860	−0.273 820	5.265 11

whence we conclude that

$$\nu_{\parallel} = 0.817\,33 \pm 0.000\,05. \tag{8}$$

For the correction to scaling exponent we estimate

$$\Omega = 1.5 \pm 0.1. \tag{9}$$

The value of $\nu_{\parallel}$ clearly rules out the simple fraction $9/11 = 0.818\,181\ldots$. The next simple rational approximant is $85/104$. The possible appearance of a large denominator in this simple problem is somewhat unexpected. But we note even larger denominators have been conjectured by Baxter and Guttman [10] and Essam *et al* [11] in critical exponents of the related directed percolation problem.

Absence of a $(1/N)$ correction in (4) is also unexpected. The apparent faster than $(1/N)$ convergence of critical exponents in some phenomenological renormalisation calculations has been ascribed to mutual cancellation between the dominant and subdominant correction to scaling terms [12]. In principle, our fitting form (6) having two correction to scaling terms can take account of such cancellations. However, fixing $A_4 = 1$ leads to distinctly poorer convergence in the five-term sequential fits than in table 3. Similarly, allowing for a constant background term in λ_N^{-1} corresponds to fixing $A_4 = 1.6346$. This also leads to poorer convergence. Note that the correction to scaling exponent for the discrete-time dynamics (directed lattice animals) may well be different from the continuum-time dynamics studied here.

I thank M Barma for useful discussions.

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